AN ULTRASONIC ANGLE BEAM METHOD FOR DETERMINING THIRD ORDER ELASTIC CONSTANTS VIA ACOUSTOElasticITY MEASUREMENTS

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ABSTRACT. Ultrasonic measurements of acoustoelastic constants, which relate wave speed changes to applied loads, are one of the few methods for experimentally determining third order elastic constants (TOECs). A typical set of measurements consists of multiple experimental setups using both longitudinal and shear transducers with directions of propagation both parallel and perpendicular to the direction of applied uniaxial stress. Experiments are generally tedious, time-consuming, and require access to both sides of a specimen, and can be problematic when the direction of propagation is parallel to the loading direction. Here we consider an angle beam technique where two transducers are mounted on the same side of a specimen with parallel sides. The general theory of acoustoelasticity is specifically considered for longitudinal and shear wave propagation at an angle to the principal stress directions for homogeneous and isotropic materials. The forward problem of calculating the acoustoelastic constants from the TOECs is considered as well as the inverse problem of determining the TOECs from measured acoustoelastic constants. Numerical results are given which show the sensitivity of the inverse calculations to small variations in the experimental measurements. Experimental data are shown for one measurement configuration and are compared to theoretical calculations using previously determined TOECs for 7075 aluminum.

Keywords: Acoustoelasticity, Third Order Elastic Constants, Ultrasonic Angle Beam

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INTRODUCTION

The acoustoelastic effect refers to the change in the speed of propagation of an elastic wave as a function of applied load, and is directly related to the third order elastic constants (TOECs) of the material. An acoustoelastic constant $K$ can be defined for a particular experiment as,

$$\frac{\Delta c}{c} = K\sigma,$$

where $c$ is the wave speed and $\sigma$ is the applied stress. The theory of acoustoelasticity was first developed by Hughes and Kelly [1] in 1953, and Pao et al. published an extensive review in 1984 [2].
Measurements of wave speed as a function of stress are one of the few means of determining TOECs, which describe non-linear deviations of the stress-strain constitutive relationship of a material. This non-linearity has also been studied by measuring higher order harmonics resulting from sinusoidal excitations, and efforts have been made to correlate these harmonics to damage; e.g. [3,4]. Direct measurement of various acoustoelastic constants has also been considered for damage detection [5,6]. For both types of measurements, the non-linear effects are small, measurements are very tedious and time-consuming, and repeatability can be a problem.

In this paper we consider determining the TOECs of an isotropic solid from angle beam acoustoelasticity measurements. This approach differs from that described in the literature in that the direction of propagation is not along any of the principal stress directions. Even though the general theory of acoustoelasticity applies to this situation, the specific theory for determining the TOECs from angle beam measurements of acoustoelastic constants has not been reported.

There are several possible advantages of an angle beam technique. First, it can readily be implemented as a one-sided technique, not requiring transducers to be mounted on both sides of a specimen. Second, it is not necessary to embed transducers inside of the specimen grips to generate beam paths along the loading direction. Third, for thin specimens, the through-thickness beam paths are much longer and the time shifts are also bigger, making the measurements more robust. And last, mounting and alignment of transducers, although still critical, is considerably simplified, particularly for thin specimens.

**THEORY**

Considered here is the theory for both the forward and the inverse problems. The forward problem is calculation of the acoustoelastic constants and expected time shifts from the applied load and material parameters, including both second and third order elastic constants. The inverse problem is determination of the TOECs from the measured time shifts given the applied load and other material parameters. For both cases, the transducers are assumed to be permanently attached to the specimen.

**Forward Problem**

The forward problem was considered by Mi et al. for angle beam shear waves [7], and the approach taken in their paper is followed here. For shear vertical (SV) waves propagating at an angle $\theta$, as shown for the angle beam configuration illustrated in Fig. 1, the effective acoustoelastic constant $K^{SV}(\theta)$ due to applied uniaxial stress is of the form,

$$\frac{\Delta c^{SV}}{c^{SV}} = [K_{1}^{SV} \sin^{2} \theta + K_{2}^{SV} \cos^{2} \theta] \sigma = K^{SV}(\theta)\sigma,$$

where $c$ is the wave speed, $\sigma$ is the applied uniaxial stress, $\theta$ is the refracted angle, and $K_{1}^{SV}$ and $K_{2}^{SV}$ are the acoustoelastic constants corresponding to $\theta = 90^\circ$ (horizontal propagation direction) and $\theta = 0^\circ$ (vertical propagation direction), respectively. The constants can be derived as [7],

$$K_{1}^{SV} = \frac{4 \mu(\lambda + \mu) + \mu m + \lambda n / 4}{2 \mu^{2}(3 \lambda + 2 \mu)} \quad \text{and} \quad K_{2}^{SV} = \frac{\mu(\lambda + 2 \mu) + \mu m + \lambda n / 4}{2 \mu^{2}(3 \lambda + 2 \mu)}.$$
FIGURE 1. Sample of thickness $h$ with a single “V” angle beam path of refracted angle $\theta$.

If the corresponding derivation is followed for longitudinal (L) waves, Eq. (2) is also valid (with a change in notation from SV to L), and the corresponding acoustoelastic constants are:

$$K_l^j = \frac{4 \lambda^2 + 15 \lambda \mu + 10 \mu^2 + 21 \mu + 4 m \lambda + 4 m \mu}{2 \mu (3 \lambda + 2 \mu) (\lambda + 2 \mu)}$$

and

$$K_t^j = \frac{-\lambda^2 - 2 \lambda \mu - m \lambda + l \mu}{\mu (3 \lambda + 2 \mu) (\lambda + 2 \mu)}.$$  

In Eqs. (3) and (4), $l$, $m$, and $n$ are the Murnaghan third order elastic constants [8], and $\lambda$ and $\mu$ are the Lamé constants. For a given loading and refracted angle, changes in both the SV and L wave speeds can thus be calculated.

The measured change in time-of-flight for a given angle beam configuration and applied load depends upon both the change in wave speed and the change in path due to deformation (i.e., the strain). For a single V path configuration (also referred to as a single skip), the resulting change in time of flight, $\Delta TOF$, can be shown to be [7],

$$\Delta TOF = \left[ \frac{2(D^2 - vh^2)}{c \sqrt{D^2 + h^2}} \right] \frac{\sigma}{E} - \left[ \frac{2K(\theta) \sqrt{D^2 + h^2}}{c} \right] \sigma,$$

where $2D$ is the transducer separation, $h$ is the sample thickness, $E$ is Young’s modulus, and $\nu$ is Poisson’s ratio. The first term is the change in time-of-flight due to deformation, and the second is that due to the change in wave speed. Similar equations can be derived for various numbers of V paths.

**Inverse Problem**

The goal of the inverse problem is to determine the three TOECs from the time-of-flight vs. load measurements. The approach to doing so can best be understood by examining Eqs. (3) and (4), which express the acoustoelastic constants in terms of the second and third order elastic constants. Note that Eq. (3) for the SV waves contains only the two constants $m$ and $n$, and Eq. (4) for the L waves contains only $l$ and $m$. To determine all three constants, three measurements are required, and they cannot be all SV or all L. An additional restriction is that there must be one SV and two L measurements because the TOECs in Eq. (3) for the SV constants have the same weighting for both coefficients. Furthermore, the two L measurements must be at two different refracted angles.
Once the configurations for the three measurements have been determined, the next step is to measure the corresponding $\Delta TOF$s for a specified applied load of $\sigma$. The three acoustoelastic constants $K(\theta)$ for the three measurement configurations can be calculated using Eq. (5). Then, Eqs. (2) and (3) for SV waves, and Eqs. (2) and (4) for L waves, can be combined to yield a set of three linear equations,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$  

The $A_{ij}$ and $B_j$ coefficients are functions of $\lambda, \mu, \theta, D$ and the acoustoelastic constants determined from the three time-of-flight measurements. This linear system of equations can then be solved for the three TOECs $l, m$ and $n$.

**EXPERIMENTAL PROCEDURE**

The acoustoelastic measurement process consists of measuring time shifts of received ultrasonic echoes as a function of the applied uniaxial load for a specific angle beam configuration. Figure 2(a) shows the measurement setup, which includes a loading fixture designed to achieve uniaxial tensile loading of the specimen via a manual hydraulic pump and cylinder. The assembly is fitted with a pressure gauge, which has been calibrated to determine the applied load. Two 5 MHz angle beam transducers with wedges designed to generate the desired refracted angle are clamped onto one side of the specimen after being coupled with light weight oil. The transducer pair is positioned and aligned based upon calculated distances for the mode, refracted angle, and number of “V” paths required. A conventional spike-mode pulser-receiver is used in through transmission mode, and the received pulses are displayed on a digital oscilloscope.

The time shift measurements were made by monotonically loading the specimen over a certain span of pressure and manually recording the actual load and resulting time shift relative to the unloaded state at each load increment. The accuracy of the time measurements is estimated to be $+/-$ 2 ns, and the accuracy of the load measurements is approximately $+/-$ 10 MPa. Typical waveforms are shown in Fig. 2(b) at both zero load and at the maximum load of 194 MPa for SV waves with a double V path.

**FIGURE 2.** (a) Angle beam measurement setup showing the loading fixture, transducers and aluminum sample. (b) Typical SV waveforms at zero load and the maximum load (double V configuration).
RESULTS

The process of determining the TOECs was evaluated numerically by ensuring that constants were exactly recovered from calculated times-of-flight. Next, a sensitivity study was performed to evaluate the effect of measurement errors. Last, preliminary experimental results are reported for one setup to verify that measured time shifts are reasonable for a material with known TOECs.

For both the numerical and experimental investigations, the specimen was a 7075 aluminum coupon with a thickness of 6.32 mm and a width of 76 mm. The reference load considered was 194 MPa, which is well below the yield strength of the material. The Murnaghan constants $l$, $m$ and $n$ were previously characterized for 7075-T651 by Stobbe [9] using conventional ultrasonic methods (i.e., beam paths perpendicular to the loading direction) and were determined to be -252.2 GPa, -325 GPa and -351.2 GPa, respectively; these are the values used here. The shear and longitudinal wave speeds were taken to be 3076 m/s and 6207 m/s, respectively, and a value of 2800 kg/m$^3$ was used for the density [9]; the Lamé constants $\lambda$ and $\mu$ were calculated from these values.

**Numerical Results**

The inversion process was first simulated using one SV and two L configurations as summarized in Table 1. These configurations were selected to mimic a proposed experimental implementation, but other possibilities are certainly valid. Times-of-flight were computed based upon a load of 194 MPa using the procedure outlined in the theory section and with constants for 7075-T651 given above. The inversion procedure of Eq. (6) was then successfully applied to exactly recover $l$, $m$ and $n$.

When there are no errors in the time-of-flight and load measurements, the inversion is perfect (within computational accuracies). In reality, there are experimental errors in the times-of-flight. Here we consider errors in the form of additive white noise with a standard deviation of +/- 1 ns and their effect on recovered TOECs. Figure 3(a) shows 1000 iterations of simulated times-of-flight where only the 2V-SV data has measurement errors. Figure 3(b) illustrates the resulting errors in the TOECs; the standard deviations for each constant are indicated in the figure. It is not surprising that only $n$ has errors because the two L-wave measurements, which do not have any additive noise, are sufficient to perfectly recover $l$ and $m$. Similarly, Fig. 4 considers errors in all of the measurements and Fig. 5 in just one of the L-wave measurements. As expected, the worst case is when all measurements have errors, but it is interesting to note from Fig. 5 that errors in only one L-wave measurement cause errors in all three TOECs.

<table>
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<th>Setup</th>
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TABLE 1. Summary of three angle beam measurement configurations.
FIGURE 3. (a) Simulated time-of-flight data for three transducer configurations where the 2V-SV values are contaminated with additive noise ($\sigma = 1$ ns). (b) Recovered TOECs using the numerical data of (a).

FIGURE 4. (a) Simulated time-of-flight data for three transducer configurations where all values are contaminated with additive noise ($\sigma = 1$ ns). (b) Recovered TOECs using the numerical data of (a).

FIGURE 5. (a) Simulated time-of-flight data for three transducer configurations where the 3V-L values are contaminated with additive noise ($\sigma = 1$ ns). (b) Recovered TOECs using the numerical data of (a).
Experimental Results

Preliminary experimental results are presented here for the forward problem of determining times-of-flight from known material properties. Results are shown in Fig. 6 for the 2V-SV configuration, and the experimental measurements agree reasonably well with the theoretical predictions. Longitudinal wave measurements are ongoing with the numbers of V paths to use for the two measurements being evaluated.

DISCUSSION

The numerical results confirm the validity and self-consistency of the proposed procedure of determining TOECs from measured load-dependent time shifts. These results also show that the resulting constants are very sensitive to experimental errors in measuring the time shifts. The solution proposed here for the inverse problem is deterministic in that three measurements are required to determine the three constants, but there is considerable freedom in selecting the three measurement configurations. Additional numerical experiments should be done to determine which configuration shows the least sensitivity to measurement errors. An alternative approach is to take more than three measurements (e.g., two shear and two longitudinal) so that the problem is over-determined and can be posed as an optimization problem. This approach would reduce the effect of measurement errors at the expense of experimental complexity.

The experimental results, although they are preliminary and consist of only one angle beam transducer configuration, are promising in that they show good agreement with the theory and thus confirm the theoretical approach for SV waves. Additional measurements are required to further validate the proposed method.

SUMMARY AND CONCLUSIONS

A method for determining third order elastic constants from a series of angle beam measurements has been proposed and the theory developed. Numerical results demonstrate the validity of the method, which is also partially confirmed by preliminary experimental measurements. This method offers several advantages over conventional acoustoelastic measurements, which include one-sided access, easier transducer setup and alignment, and the potential for in situ measurements for possible monitoring of damage. Future work should concentrate on additional numerical simulations to help determine the optimum transducer configuration combined with experimental verification of the overall approach.

FIGURE 6. Experimental and theoretical time shifts vs load for the shear double “V” configuration.
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REFERENCES