The inverse source problem for an oblique force on an elastic plate

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A time-dependent concentrated force applied obliquely on the surface of a plate generates elastic waves in the plate. The determination of the location, orientation, and time history of the force from the transient wave records is referred to as the inverse source problem of elastic waves. This paper presents an iterative method of deconvolution which determines the orientation and time-dependent amplitude of the force from the transient response of the plate surface at a minimum of two locations, the source location being given. Numerical results are presented for forces with various orientations and time histories, and for synthetic data both with and without noise.

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INTRODUCTION

A concentrated force applied to the surface of an infinite plate for a finite period of time generates transient wave motion. If the direction of the force and its time history are known, the nearfield response can be calculated by the method of generalized rays, and the farfield response by the method of normal modes. This paper describes a method to solve an inverse problem; that is, to determine the orientation and time-dependent amplitude of the force from the transient displacement signals which are measured at a discrete number of receiver locations.

Inverse problems of a similar nature have long been in existence. In this paper by Goodier et al., an integral equation was solved to determine the source time function of a vertical force on the surface of a half-space from the farfield response. In the paper by Hsu et al., a time convolution integral was discretized and inverted to determine the source time function of a vertical force from the epicentral response of a large plate. The off-center response for a vertical force on a large plate was inverted by Michaels et al., to obtain the source time function. For the work described in Refs. 5 and 6, the source time function is related to the response function by a convolution integral with a single kernel (Green's function). This type of inverse problem may be solved by one of many deconvolution procedures.

In the field of seismology, a similar problem is that of determining the components of the symmetric moment tensor characterizing a point earthquake or explosion. Dziewonski and Gilbert assumed that each component had a different time dependence, and performed a least squares minimization in the frequency domain. Stump and Johnson also assumed different time dependencies, but formulated the least squares problem in the time domain. Barber and Langston assumed that each moment tensor component had the same time function, and that this function could be described by a small number of rectangular pulses, where each pulse had the same predetermined width and unknown amplitude. They then used the generalized inverse method of Wiggins to solve the nonlinear least squares problem.

For the inverse problem considered here, that of an oblique force applied to the surface of either a half-space or plate, the response is given by the convolution of a linear combination of Green's functions with a single source time function. Thus, the single kernel deconvolution method for a vertical force cannot be directly applied. Also, the method developed for the seismic source inverse problem in Refs. 10 and 11, which assumes multiple time functions, cannot be applied here. The method in Ref. 12, where a single source time function is assumed, requires both linearization of the least squares problem and modeling of the time function by a limited number of parameters.

In this paper we present a general procedure of deconvolution for the case where the kernel of the convolution time integral is a linear combination of Green's functions with unknown coefficients. Linearization is not required, and the only restriction on the source time function is that it is of finite length. This method is then applied to the problem of an obliquely applied force, for which the unknown coefficients of the Green's functions are proportional to the direction cosines of the force, and the unknown source time function is the same for the three force components.

The inverse method described here requires knowledge of the Green's functions of the medium. We assume that the position of the source is known so that the Green's functions which relate the response to the source can be accurately calculated. The problem of source location for acoustic emission was reviewed by Pao, and calculation of the Green's functions for a plate is discussed in Refs. 2 and 3.

I. FORWARD PROBLEM

The displacement field generated by a point force in or on a plate can be determined from a convolution integral of the forcing function and the Green's displacement tensor, which is the displacement in the $i$th direction at $x$ and $t$ due to an impulsive concentrated force of unit magnitude in the $i$th direction at $x_0$ and $t_0$. Thus, for a point force $F(x_0,t)$ at $x_0$ that is zero for $t_0<0$, the resulting displacement is

$$u_i(x,t) = \sum_{j=0}^{3} \int_{j=0}^{3} G_{ij}(x,t,x_0,0) F_j(x_0,t) \, d\tau$$

In this and subsequent equations, an asterisk denotes the convolution integral in the time variable. If the orientation of
the force at \( x_0 \) is independent of time, \( F \) may be expressed as
\[
f(x_0, t) = f(0, t),
\]
where \( f \) is a unit vector. The previous equation is written as
\[
u_i(x, t) = \left( \sum_{i=1}^{3} G_j(x, t; x_0, t) f_j(x_0) \right) * s(t).
\]
(2)

Thus the displacement in the \( i \)th direction is the weighted sum of three Green's functions convolved with a single source time function \( s(t) \).

In cylindrical coordinates \( x = (r, \theta, z) \), as shown in Fig. 1, the source acts at \( x_0 = 0 \) and the receiver is situated on the plane \( \theta = 0 \). The Cartesian components of the displacement vector \( u_i(r, 0, z, t) \) are given by
\[
u_i(r, 0, z, t) = \left( \sum_{j=1}^{3} f_j(0) G_j(r, 0, z, t; 0, 0) \right) * s(t).
\]
(3)
The \( G_j(r, 0, z, t; 0, 0) \) are the Cartesian components of the Green's tensor for an infinite plate evaluated at \( \theta = 0 \), due to a source at \( x_0 = 0 \), and \( t_0 = 0 \).

For a receiver at an angle \( \theta \) but at the same \( r \) and \( z \), we first rotate the \( x_i \) and \( x_j \) axes about the \( x_0 \) (or \( z \)) axis to \( x'_i \) and \( x'_j \) through the angle \( \theta \) (Fig. 1), such that the receiver is directly above the \( x'_i \) axis. We calculate the components of \( f \) in the primed coordinate system by a coordinate transformation:
\[
\begin{align*}
f'_1 &= f_1 \cos \theta + f_2 \sin \theta, \\
f'_2 &= -f_1 \sin \theta + f_2 \cos \theta, \\
f'_3 &= f_3.
\end{align*}
\]
(4)

Since the medium under consideration is axially symmetric, we may now apply Eq. (3) to evaluate the Cartesian components of the displacement in the primed system, \( u'_i(r, 0, z, t) \) \( (i = 1, 2, 3) \):
\[
u'_i(r, 0, z, t) = \left( \sum_{j=1}^{3} f'_j(0) G_j(r, 0, z, t; 0, 0) \right) * s(t).
\]
(5)
The Green’s functions in Eq. (5) are the same as those in Eq. (3). Thus, if the medium is axially symmetric and the source is located on the symmetry axis, the same Green’s functions are used to calculate displacement at any angle \( \theta \) for a given \( r \) and \( z \).

The displacement in the unprimed system is calculated from the displacement in the primed system by the inverse of the transformation in Eq. (4):
\[
u_i(r, \theta, z, t) = u'_i(r, 0, z, t) \cos \theta - u'_2(r, 0, z, t) \sin \theta, \\
u_2(r, \theta, z, t) = u'_1(r, 0, z, t) \sin \theta + u'_2(r, 0, z, t) \cos \theta, \\
u_3(r, \theta, z, t) = u'_3(r, 0, z, t).
\]
(6)

II. INVERSE SOURCE PROBLEM

The goal of this study is to determine \( f(x_0) \) and \( s(t) \) from the normal displacement \( u_3 \) measured at several receiver locations for a finite time interval, assuming that the source location \( x_0 \) is known and that the origin of the coordinate system is located at \( x_0 \). We first determine the coefficients \( f'_1 \) and \( f'_2 \) and the source time function \( s(t) \) at each receiver location, and then analyze these results to obtain the final estimates of \( f_1, f_2, f_3 \), and \( s(t) \).

A. Deconvolution with multiple Green’s functions

Consider the following convolution representation:
\[
u(t) = \left( \sum_{i=1}^{K} c_i G_i(t) \right) * s(t).
\]
(8)

Equation (7) has this form with \( u = u_3, \ R = 2, \ c_1 = f'_1, \ c_2 = f'_2, \ G_1(t) = G_3(r, 0, z, t; 0, 0) \), and \( G_2(t) = G_3(r, 0, z, t; 0, 0) \). The problem of deconvolution is to simultaneously determine \( s(t) \) and the \( c_i \), given \( u(t) \) and the \( G_i(t) \) over a finite time interval. Note that the problem is nonlinear because the \( c_i \) and \( s(t) \) appear in the form of a product. However, if either the \( c_i \) or \( s(t) \) are known, the other can be estimated by a linear inverse procedure. The two linear subproblems of finding \( s(t) \) and the \( c_i \) separately are solved in this paper by the least squares method.

We first discretize the time variable by assuming that the time functions are sampled at an interval of \( \Delta t \). The continuous time index \( t \) is replaced by \( n \), which corresponds to the time at \( (n - 1)\Delta t \). Let \( N \) be the value of \( n \) which corresponds to the maximum time for which \( u(t) \) and \( G(t) \) are specified. The convolution integral in Eq. (8) reduces to a summation, where a constant factor \( \Delta t \) has been dropped:
\[
u(n) = \sum_{k=1}^{n} G_k(n - k + 1) s(k), \quad n = 1, 2, \ldots, N.
\]
(9)

Consider first the problem of determining the \( \tilde{c}_i \), estimates of the \( c_i \), where \( \tilde{s}(n) \) is given as an estimate of \( s(n) \). Let \( \tilde{u}(n) \) be the estimate of \( u(n) \) based on the \( \tilde{c}_i \) and \( \tilde{s}(n) \):
\[
\tilde{u}(n) = \sum_{i=1}^{K} \tilde{c}_i \sum_{k=1}^{n} G_i(n - k + 1) \tilde{s}(k).
\]
(10)

If \( q_i(n) \) is the convolution of \( G_i(n) \) with \( \tilde{s}(n) \), we have
\[
q_i(n) = \sum_{k=1}^{n} G_i(n - k + 1) \tilde{s}(k)
\]
(11)
and Eq. (10) reduces to

$$\hat{u}(n) = \sum_{i=1}^{R} \tilde{c}_i q_i(n).$$  \hspace{1cm} (12)

The mean square error $E$ between the $N$ sample points of $u(n)$ and $\hat{u}(n)$ is

$$E = \sum_{n=1}^{N} \left( u(n) - \sum_{i=1}^{R} \tilde{c}_i q_i(n) \right)^2.$$  \hspace{1cm} (13)

The conditions $\partial E / \partial \tilde{c}_k = 0$ give rise to a system of equations for the $\tilde{c}_i$:

$$\sum_{n=1}^{N} \sum_{k=1}^{R} q_i(n) q_k(n) \tilde{c}_i = \sum_{n=1}^{N} u(n) q_k(n), \quad k = 1, 2, \ldots, R.$$  \hspace{1cm} (14)

This system of $R$ equations ($k = 1, 2, \ldots, R$) can be solved for the $R$ number of coefficients $\tilde{c}_i$.

Next consider the determination of $\tilde{s}(n)$ where the $\tilde{c}_i$ are given. We assume that $\tilde{s}(n)$ is of length $K$, where $K$ is less than $N$, the number of points in $u(n)$ and the $G_i(n)$. Let $r(n)$ be the weighted sum of the $G_i(n)$ in Eq. (10):

$$r(n) = \sum_{i=1}^{K} \tilde{c}_i G_i(n).$$  \hspace{1cm} (15)

Then Eq. (10) reduces to

$$\hat{u}(n) = \sum_{k=1}^{K} r(n - k + 1) \tilde{s}(k).$$  \hspace{1cm} (16)

The mean square error $E$ is

$$E = \sum_{n=1}^{N} \left( u(n) - \sum_{k=1}^{K} r(n - k + 1) \tilde{s}(k) \right)^2.$$  \hspace{1cm} (17)

The conditions $\partial E / \partial \tilde{s}(j) = 0$ give rise to a system of equations:

$$\sum_{n=1}^{N} \sum_{k=1}^{K} r(n - j + 1) r(n - k + 1) \tilde{s}(k)$$

$$= \sum_{n=1}^{N} u(n) r(n - j + 1), \quad j = 1, 2, \ldots, K.$$ \hspace{1cm} (18)

This system of $K$ equations ($j = 1, 2, \ldots, K$) can be solved for the $K$ samples of $\tilde{s}(n)$. When evaluating the summations, we must use the information that $\tilde{s}(n) = 0$ for $n > K$ and $r(n) = 0$ for $n < 1$.

This method for determining $\tilde{s}(n)$ is similar to the least squares method described by Robinson in Ref. 8. The main difference is that here $u(n)$ and $r(n)$ are given for a finite time interval because the Green's functions are available for only a finite time interval. This same problem was considered by Symes in Ref. 9. However, in the method described by Robinson, it is assumed that both the source function and the convolution kernel are of finite length, and that $u(n)$, which is now also of finite length, is completely specified. Thus, the system of equations for $\tilde{s}(n)$ has a special form and fast solution methods can be used. In our case, the system of equations has no special form, and fast solution methods such as fast Fourier transform division and Toeplitz recursion cannot be used.

An iterative algorithm to determine both the $\tilde{c}_i$ and $\tilde{s}(n)$ in Eq. (10) has been developed on the basis of these two linear subproblems. First, the $\tilde{c}_i$ coefficients are set to arbitrary nonzero initial values. Using these $\tilde{c}_i$, the $\tilde{s}(n)$ are calculated from Eq. (18). Next, these $\tilde{s}(n)$ are used to calculate improved $\tilde{c}_i$ from Eq. (14). This procedure of alternately calculating $\tilde{s}(n)$ and the $\tilde{c}_i$ can continue until one of several possibilities occurs:

(i) The $\tilde{c}_i$ and $\tilde{s}(n)$ converge to fixed values (possibly incorrect) in a reasonable number of iterations.

(ii) The $\tilde{c}_i$'s and $\tilde{s}(n)$ are converging very slowly, but it does not appear that convergence will be reached in a reasonable number of iterations.

(iii) There is no evidence of convergence.

Thus far, a general theorem for convergence is still not available. Conditions and rate of convergence were recently investigated by Symes,16 who showed that convergence does occur for most initial values of the $\tilde{c}_i$. Our experience is that it is readily apparent which of the above possibilities is occurring by observing the $\tilde{c}_i$, $\tilde{s}(n)$, and the error after each iteration. If convergence is reached, the error stabilizes and the $\tilde{c}_i$ and $\tilde{s}(n)$ remain constant. If convergence is slow, the error changes slightly with each iteration and the $\tilde{c}_i$ and $\tilde{s}(n)$ also change slightly. If there is no convergence, the $\tilde{c}_i$ and $\tilde{s}(n)$ change significantly even though the error changes only slightly.

For optimum convergence, Eqs. (14) and (18) for the $\tilde{c}_i$ and $\tilde{s}(n)$ should be well conditioned.15 Thus, the $G_i(n)$ should be orthogonal, $s(n)$ should be broadband, and the length of $\tilde{s}(n)$ should be short compared to the length of $u(n)$ ($K < N$). Numerical testing of this algorithm supports this statement.

Because the $\tilde{c}_i$ and $s(n)$ are present in Eq. (9) as a product, they are defined only to within a scale factor. To uniquely define them, we scale $\tilde{s}(n)$ and the $\tilde{c}_i$ after each iteration such that $\tilde{s}(n)$ has positive mean and unity rms value:

$$\sum_{n=1}^{K} \tilde{s}(n) > 0,$$  \hspace{1cm} (19)

$$\frac{1}{K} \sum_{n=1}^{K} \tilde{s}(n)^2 = 1.$$  \hspace{1cm} (20)

Thus, $\tilde{s}(n)$ and the $\tilde{c}_i$ are unambiguously determined.

B. Determination of ($f_1, f_2, f_3$) and $s(t)$

The deconvolution procedure yields estimates of the $R$ coefficients $\tilde{c}_i$ and the $K$ samples of $s(t)$ at each of the $M$ number of receiver locations. Let $\tilde{c}_{m}^n$ and $\tilde{s}_{m}^n(n)$ be the estimated coefficients and source time function, respectively, at the $n$th receiver, which is located at $r = r_{m}$ and $\theta = \theta_{m}$. Of course, the $\tilde{s}_{m}^n(n)$ should be the same at each receiver, but they may differ because of noise in the data and numerical errors in the deconvolution. An improved $\tilde{s}(n)$ is obtained by averaging over the $M$ receivers:

$$\tilde{s}(n) = \frac{1}{M} \sum_{m=1}^{M} \tilde{s}_{m}^n(n).$$  \hspace{1cm} (21)

A weighted average is preferable if the data at some receivers are noisier than at others.

The coefficients at each receiver are functions of the direction cosines. The coefficient $\tilde{c}_{m}^n$ is proportional to $f_1^m(\theta_{m})$ and $\tilde{s}_{m}^n(n)$ is proportional to $f_1^m(\theta_{m})$ according to Eqs. (4), (7), and (8). We define a constant of proportionality $K$ such that
\[ \hat{c}^m_1 = f_1' K = (f_1 \cos \theta_m + f_2 \sin \theta_m) K, \]
\[ \hat{c}^m_2 = f_1' K = f_1 K. \]  
(22)

The constant \( K \) is determined from the condition that \( f \) is a unit vector.

Estimates of \( Kf_1 \) and \( Kf_2 \) are obtained by fitting the \( \hat{c}^m_1 \) data to the sinusoidal form of Eq. (22). The following system of equations is obtained for \( Kf_1 \) and \( Kf_2 \):
\[
\begin{bmatrix}
\sum_{m=1}^{M} \cos^2 \theta_m & \sum_{m=1}^{M} \sin \theta_m \cos \theta_m \\
\sum_{m=1}^{M} \sin \theta_m \cos \theta_m & \sum_{m=1}^{M} \sin^2 \theta_m
\end{bmatrix}
\begin{bmatrix}
Kf_1 \\
Kf_2
\end{bmatrix}
= \begin{bmatrix}
\sum_{m=1}^{M} \hat{c}^m_1 \cos \theta_m \\
\sum_{m=1}^{M} \hat{c}^m_1 \sin \theta_m
\end{bmatrix}
\]  
(23)

Thus, the angular dependence of the \( \hat{c}^m_1 \) coefficient yields both horizontal components of the force.

Since Eq. (23) contains two unknowns, there must be at least two receivers to determine both \( Kf_1 \) and \( Kf_2 \). More receivers will provide redundancy and reduce errors due to noise. The receivers must be located at different angular positions, but can be at arbitrary radial distances.

The vertical component is obtained by averaging the \( \hat{c}^m_2 \) data:
\[ Kf_3 = \frac{1}{M} \sum_{m=1}^{M} \hat{c}^m_2. \]  
(24)

Note that \( Kf_3 \) can be obtained from just one receiver.

The direction cosines are found by normalizing the vector \( Kf \):
\[ f = K f/|K f|. \]  
(25)

This normalization is used to rescale \( \hat{f}(n) \):
\[ \hat{f}(n) = \hat{f}(n)|K f|. \]  
(26)

The entire procedure for calculating \( f_1, f_2, f_3, \) and \( \hat{f}(n) \) is thus completed.

It is possible to improve these estimates by using the information that the source function is the same at all of the receivers and that the coefficients at the different receivers are related by Eq. (22). For example, the fitted direction cosines could be used to solve for improved source functions at each receiver location. Or the least squares problem could be formulated such that the source function is forced to be the same at each receiver. There are many different iteration schemes and variations of least squares that, if necessary, could be implemented to improve results.

III. NUMERICAL RESULTS

Numerical experiments were performed to evaluate this inverse algorithm. First, displacements were calculated at several receiver locations for known sources. Second, the direction cosines and source time function were determined by the inverse method. Third, the results obtained by inversion were compared to the known values.

All Green’s functions were calculated by a computer program developed at Cornell University by R. Gajewski and A. Ceranoglu. This program is based upon the generalized ray theory, which provides an exact solution for transient waves in a plate of finite thickness that is infinite in extent. In this theory, which is described in Refs. 1 and 2, the elastic waves that propagate along various ray paths are represented by a series of ray integrals. For a specific recording time, only a finite number of integrals contribute to the solution, and these integrals are evaluated numerically. Thus, the Green’s functions obtained in this manner are very accurate.

A. Calculation of displacements

The normal displacement signals \( u_z(x, \theta, x, t) \) were calculated for forces located at \( x_0 = (0, 0, 0) \) and receivers located at \( x = (2h, 0, h, t) \), where \( h \) is the thickness of the plate. Note that the receivers are on the opposite side of the plate as the source, as shown in Fig. 1. For this geometry, the two Green’s functions \( G_{33}(2h, 0, h, t, 0, 0) \) and \( G_{33}(2h, 0, h, t, 0, 0) \) are needed, and are shown in Fig. 2. For these numerical tests, 115 time points were calculated for each Green’s function. The units of time in this and subsequent figures are normalized by the time for a longitudinal wave to travel one plate thickness, \( T = h/c \).

Three source time functions were used to calculate displacements. They are a rectangular pulse, a cosine bell, and a sawtooth, and are shown in Fig. 3. Each of these functions consists of 25 time points. Three orientations for the applied force were also used with these source time functions—a vertical force, a horizontal force, and an oblique force. The direction cosines, or the Cartesian components of these forces with unit magnitude, are listed in Table I, which summarizes the five cases for which the vertical displacement was calculated. For all cases, 115 time points were calculated at four receivers located at \( \theta = 0^\circ, 90^\circ, 150^\circ, \) and \( 285^\circ \).

The displacement response for a point force is given by Eq. (7). For a vertical force, the displacement is proportional to the convolution of the source time function with \( G_{33} \). There is no angular dependence and the displacement is the same at all of the receivers since they are at a common radius. For a horizontal force, the displacement is the convolution

![Graphs](https://example.com/graphics.png)

**FIG. 2.** Cartesian components of the Green’s displacement tensor at \( x(x, \theta, z) = (2h, 0, h) \) for a source at \( x_0 = (0, 0, 0) \). The normalized time is \( c_L \).


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of the source time function with \( G_{31} \) multiplied by \( f_1 \), which does depend upon \( \theta \). Thus, the same time function is measured at each receiver but with a \( \theta \)-dependent amplitude.

For an oblique force source, both \( G_{31} \) and \( G_{33} \) in Eq. (7) contribute to the response, and different displacement signals are measured at each receiver. This can be seen in Fig. 4, which shows the displacement signals for case 5 in Table I, for which the source is an oblique force with a sawtooth source time function.

**B. Calculation of \( (f_1, f_2, f_3) \) and \( \hat{s}(t) \)**

The deconvolution method was applied to the displacement waveforms at each receiver for each case in Table I. For each step of the iteration, Gaussian elimination was used to solve Eqs. (14) and (18). The coefficients \( \hat{c}_1^n \) and \( \hat{c}_2^n \) were both initially set to one. For all waveforms, convergence to the correct coefficients and source function with accuracy to four significant digits was obtained in no more than 20 iterations. Figure 5 shows typical estimated source functions after 1, 10, 13, and 20 iterations. They were obtained from the response at \( \theta = 0 \) for case 4. The values for \( \hat{c}_1^n \) and \( \hat{c}_2^n \) at each iteration are also listed in the figure, where they have been multiplied by \( \sqrt{5} \).

The \( G_{31} \) coefficients \( \hat{c}_1^n \) were fitted to a sinusoidal wave, and the \( G_{33} \) coefficients \( \hat{c}_2^n \) were averaged to obtain estimates of the direction cosines according to Eqs. (23), (24), and (25). The recovered direction cosines after 20 iterations are listed in Table II, and show excellent agreement with the exact values in Table I. The recovered source functions after 20 iterations are identical to the exact ones in Fig. 3 within plotting accuracies, and are not separately shown.

**TABLE I. Parameters for displacement calculations.**

<table>
<thead>
<tr>
<th>Case number</th>
<th>( (f_1, f_2, f_3) )</th>
<th>( s(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0,0,1)</td>
<td>rectangular pulse</td>
</tr>
<tr>
<td>2</td>
<td>(1,1,0)/( \sqrt{5} )</td>
<td>rectangular pulse</td>
</tr>
<tr>
<td>3</td>
<td>(2,0, -1)/( \sqrt{5} )</td>
<td>rectangular pulse</td>
</tr>
<tr>
<td>4</td>
<td>(2,0, -1)/( \sqrt{5} )</td>
<td>cosine bell</td>
</tr>
<tr>
<td>5</td>
<td>(2,0, -1)/( \sqrt{5} )</td>
<td>sawtooth</td>
</tr>
</tbody>
</table>

**C. The effect of noise**

The stability of this inverse method in the presence of noise was tested by adding white noise to the displacement data. This was done for case 5, and the noisy displacements are shown in Fig. 6 in contrast to the clean data in Fig. 4. The standard deviation of the noise is about 3% of the peak displacement amplitude.

The direction cosines and the source time function were estimated by the inverse method, and the results are shown in Fig. 7. The recovered source function is a reasonable estimate of the exact one, with the average absolute error amplitude about 7.5% of the average absolute signal amplitude. The orientation of the source, as given by the direction cosines, is recovered with only a 0.5° error.

**IV. DISCUSSION AND CONCLUSIONS**

In this paper we have presented an inverse method for obtaining the direction and time-dependent amplitude of a force acting on the surface of an infinite plate. Although no general mathematical proof is available for the convergence
of this iterative inversion procedure, the method is computationally accurate for realistic situations.

The three source time functions considered here range from narrow bandwidth (cosine bell) to wide bandwidth (rectangular pulse). The deconvolution step of the inverse method successfully recovered all three source functions. As expected, the narrow-band cosine bell required a few more iterations than the wideband rectangular pulse.

For an oblique force, a linear combination of Green’s functions appears in the convolution, whereas there is only one Green’s function for a horizontal or vertical force. The inversion method successfully recovered the direction cosines and source time function for all three cases, even when one coefficient was zero.

The inversion method yielded good results in the presence of noise for the one example considered here. If necessary, the algorithm is easily modified to handle statistical noise similar to the procedure described by Robinson.17

In this paper we have assumed a unit magnitude for the applied force and used the calculated displacement, $u_3$, as the input for the inverse method. When the procedure is applied to real data, the magnitude of the force remains unknown unless the measuring instrument has been calibrated against a force of known magnitude and orientation.

The convolution representation of Eq. (8) could have been formulated in the frequency domain. Then, the convolution would become a multiplication and all signals would be expressed as a function of frequency. The major difficulty in a frequency domain formulation is that it assumes that the measured signals have been recorded for a sufficiently long time such that they have decayed to insignificant levels. For experimentally recorded transient signals, this is seldom the case, and signals must be windowed in time before transforming to the frequency domain. One of the key features of the time domain method presented here is that no special windowing operations are necessary to compensate for signal truncation. Thus, even though frequency domain methods could be used to solve this problem, the time domain formulation is more straightforward because of the manner in which the data are recorded.

It should be noted that this inverse method is not restricted to sources of concentrated forces. It can be applied to any other type of source for which the response is given by a linear combination of Green’s functions with unknown coefficients, each convolved with a common unknown source time function, as in the form of Eq. (8). For example, two other inverse problems have been solved by applying the iterative technique presented here. The one is the problem of distributed forces or sources when the spatial variables are discretized; the other is the radiation by a crack in a solid or a fault-slip in the earth’s crust, where the source is represented by a moment density tensor with six independent components. The former was reported for a line source by Chang and Sichess,18 who applied the method of singular value decomposition to solve the linear systems of algebraic equations, Eqs. (14) and (18). A preliminary study of the latter was reported by the authors.19

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